30. Signal Space Analysis of BASK, BFSK, BPSK, and QAM

The vector-space representation of signals and the optimum detection process which chooses the signal closest to the received signal is particularly useful in signal design and in probability-of-error calculations. The material presented herein includes the error performance of binary amplitude-shift keying (BASK), binary frequency-shift keying (BFSK), binary phase-shift keying (BPSK), and $M$-ary quadrature amplitude modulation (QAM) signals.

Error Performance of Binary ASK

A binary amplitude-shift keying (BASK) signal can be defined by

$$s(t) = \begin{cases} A \cos(2\pi f_c t), & 0 \leq t \leq T \\ 0, & \text{elsewhere} \end{cases}$$

(30.1)

where $A$ is a constant, $f_c$ is the carrier frequency, and $T$ is the bit duration. It has a power $P = A^2/2$, so that $A = \sqrt{2P}$. Thus equation (30.1) can be written as

$$s(t) = \begin{cases} \sqrt{2P} \cos(2\pi f_c t), & 0 \leq t \leq T \\ 0, & \text{elsewhere} \end{cases}$$

$$= \begin{cases} \sqrt{PT} \frac{2}{\sqrt{T}} \cos(2\pi f_c t), & 0 \leq t \leq T \\ 0, & \text{elsewhere} \end{cases}$$

$$= \begin{cases} \sqrt{E} \frac{2}{\sqrt{T}} \cos(2\pi f_c t), & 0 \leq t \leq T \\ 0, & \text{elsewhere} \end{cases}$$

(30.2)

where $E = PT$ is the energy contained in a bit duration. Figure 30.1 shows the signal constellation diagram of BASK signals and the conditional probability density functions associated with the signals.

Figure 30.1 BASK signal constellation diagram and the conditional probability density functions associated with the signals.

The energy contained in a bit duration is

$$E = d^2$$

(30.3)

and so
\[ d = \sqrt{E} \]  

(30.4)

It is assumed that the noise is *additive white Gaussian noise* (AWGN) with a two-sided power spectral density of \( n_0/2 \), *zero mean* and fixed variance \( \sigma^2 = n_0/2 \). In the presence of AWGN, the conditional probability density function of \( \phi_1 \) assuming that \( s_0 \) is transmitted is

\[
f(\phi_1 / 0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\phi_1^2}{2\sigma^2}}
\]

and the probability of error given that \( s_0 \) is transmitted is

\[
P_{e0} = \int_{\phi_1 = d/2}^{\infty} f(\phi_1/0) d\phi_1 = \int_{\phi_1 = d/2}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\phi_1^2}{2\sigma^2}} d\phi_1 = P_{e1}
\]

(30.5)

Similarly, the probability of error given that \( s_1 \) is transmitted is

\[
P_{e1} = \int_{-\infty}^{\phi_1 = d/2} f(\phi_1/1) d\phi_1 = \int_{-\infty}^{\phi_1 = d/2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\phi_1-d)^2}{2\sigma^2}} d\phi_1 = P_{e1}
\]

where

\[
f(\phi_1 / 1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\phi_1-d)^2}{2\sigma^2}}
\]

(30.6)

Let \( p_0 \) be the probability of sending \( s_0 \) and \( p_1 \) be the probability of sending \( s_1 \). For equally likely transmission of binary signals, we have \( p_0 = p_1 = 0.5 \). The average probability of error is given by

\[
P_e = p_0 P_{e0} + p_1 P_{e1} = P_{e0} = \int_{\phi_1 = d/2}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\phi_1^2}{2\sigma^2}} d\phi_1
\]

(30.7)
Let \( y = \frac{\phi_1}{\sqrt{2\sigma}} \). Then \( y^2 = \frac{\phi_1^2}{2\sigma^2} \) and \( dy = \frac{d\phi_1}{\sqrt{2\sigma}} \). Substituting \( y \) and \( dy \) into equation (30.7), we get

\[
P_e = \int_{y=\frac{d}{2\sqrt{2\sigma}}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} \, dy
\]

\[
= \frac{1}{2} \left[ \frac{2}{\sqrt{\pi}} \int_{y}^{\infty} e^{-y^2} \, dy \right]
\]

\[
= \frac{1}{2} \text{erfc} \left( \frac{d}{2\sqrt{2\sigma}} \right)
\]

\[
= \frac{1}{2} \text{erfc} \left( \frac{d}{2\sqrt{n_0}} \right)
\]

\[
= \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E}{4n_0}} \right)
\]

where the complementary error function is

\[
erfc \, (x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-\alpha^2} \, d\alpha
\]

\[
\sigma^2 = n_0/2 \text{ for AWGN, and } d = \sqrt{E}.
\]

**Error Performance of Binary PSK**

A binary phase-shift keying (BPSK) signal can be defined by

\[
s(t) = \pm A \cos 2\pi f_c t, \quad 0 \leq t \leq T
\]

(30.10)

where \( A \) is a constant, \( f_c \) is the carrier frequency, and \( T \) is the bit duration. It has a power \( P = A^2/2 \), so that \( A = \sqrt{2P} \). Thus equation (30.10) can be written as

\[
s(t) = \pm \sqrt{2P} \cos 2\pi f_c t
\]

\[
= \pm \sqrt{PT} \frac{2}{\sqrt{T}} \cos 2\pi f_c t
\]

\[
= \pm \sqrt{E} \frac{2}{\sqrt{T}} \cos 2\pi f_c t
\]

(30.11)
where $E = PT$ is the energy contained in a bit duration. Figure 30.2 shows the signal constellation diagram of BPSK signals and the conditional probability density functions associated with the signals.

**Figure 30.2** BPSK signal constellation diagram and the conditional probability density functions associated with the signals.

The energy contained in a bit duration is

$$E = \left(\frac{d}{2}\right)^2$$  \hspace{1cm} (30.12)

and so

$$d = \sqrt{4E}$$  \hspace{1cm} (30.13)

In the presence of AWGN, the conditional probability density function of $\phi_1$ assuming that $s_0$ is sent is

$$f(\phi_1|0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\phi_1 + \frac{d}{2})^2}{2\sigma^2}}$$  \hspace{1cm} (30.14)

and the probability of error given that $s_0$ is transmitted is

$$P_{e0} = \int_{0}^{\infty} f(\phi_1|0) d\phi_1 = \int_{0}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\phi_1 + \frac{d}{2})^2}{2\sigma^2}} d\phi_1$$  \hspace{1cm} (30.15)

Similarly, the probability of error given that $s_1$ is sent is

$$P_{e1} = \int_{-\infty}^{0} f(\phi_1|1) d\phi_1 = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\phi_1 - \frac{d}{2})^2}{2\sigma^2}} d\phi_1 = P_{e0}$$  \hspace{1cm} (30.16)

where
Let $p_0$ be the probability of sending $s_0$ and $p_1$ be the probability of sending $s_1$. For equally likely transmission of binary signals, we have $p_0 = p_1 = 0.5$. The average probability of error is

$$P_e = p_0 P_{e0} + p_1 P_{e1}$$

$$= P_{e0}$$

$$= \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(\phi_1 + \frac{d}{2})^2}{2\sigma^2}} d\phi_1$$

Let $y = \frac{\phi_1 + \frac{d}{2}}{\sqrt{2}\sigma}$. Then $y^2 = \frac{(\phi_1 + \frac{d}{2})^2}{2\sigma^2}$ and $dy = \frac{d\phi_1}{\sqrt{2}\sigma}$. Substituting $y$ and $dy$ into equation (30.18), we get

$$P_e = \int_{\phi_1 + \frac{d}{2}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} dy$$

$$= \frac{1}{2} \left[ \frac{2}{\sqrt{\pi}} \right] \int_{\phi_1 + \frac{d}{2}}^{\infty} e^{-y^2} dy$$

$$= \frac{1}{2} \text{erfc} \left( \frac{\phi_1 + \frac{d}{2}}{\sqrt{2}\sigma} \right)$$

$$= \frac{1}{2} \text{erfc} \left( \frac{d}{2\sqrt{2}\sigma} \right)$$

$$= \frac{1}{2} \text{erfc} \left( \frac{d}{2\sqrt{n_0}} \right)$$

$$= \frac{1}{2} \text{erfc} \left( \frac{E}{\sqrt{n_0}} \right)$$

(30.19)

where $\phi_1 = 0$ is the threshold value, the complementary error function is
\[ \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\alpha^2} \, d\alpha \quad (30.20) \]

\[ \sigma^2 = n_0/2 \text{ for AWGN, and } d = \sqrt{4E}. \]

**Error Performance of Orthogonal Binary FSK**

A binary frequency-shift keying (BFSK) signal can be defined by

\[
s(t) = \begin{cases} 
A \cos(2\pi f_{c1} t), & 0 \leq t \leq T \\
A \cos(2\pi f_{c2} t), & \text{elsewhere}
\end{cases} \quad (30.21)
\]

where \( A \) is a constant, \( f_{c1} \) and \( f_{c2} \) are the carrier frequencies, and \( T \) is the bit duration. It has a power \( P = A^2/2 \), so that \( A = \sqrt{2P} \). Thus equation (30.21) can be written as

\[
s(t) = \begin{cases} 
\sqrt{2P} \cos(2\pi f_{c1} t), & 0 \leq t \leq T \\
\sqrt{2P} \cos(2\pi f_{c2} t), & \text{elsewhere}
\end{cases}
\]

\[
= \begin{cases} 
\sqrt{PT} \frac{2}{T} \cos(2\pi f_{c1} t), & 0 \leq t \leq T \\
\sqrt{PT} \frac{2}{T} \cos(2\pi f_{c2} t), & \text{elsewhere}
\end{cases}
\]

\[
= \begin{cases} 
\sqrt{E} \frac{2}{T} \cos(2\pi f_{c1} t), & 0 \leq t \leq T \\
\sqrt{E} \frac{2}{T} \cos(2\pi f_{c2} t), & \text{elsewhere}
\end{cases} \quad (30.22)
\]

where \( E = PT \) is the energy contained in a bit duration. Figure 30.3 shows the signal constellation diagram of orthogonal BFSK signals and the conditional probability density functions associated with the signals.

**Figure 30.3** Orthogonal BFSK signal constellation diagram and the conditional probability density functions associated with the signals.

The energy in a bit duration is

\[ E = \left( \frac{d}{\sqrt{2}} \right)^2 \quad (30.23) \]

and
\[ d = \sqrt{2E} \]  

(30.24)

If we rotate the two orthonormal axes \( \phi_1 \) and \( \phi_2 \) to \( u_1 \) and \( u_2 \), we end up with a signal constellation diagram for BPSK signals with a separation distance of \( d \). Thus, the average probability of error is

\[
P_e = \frac{1}{2} \text{erfc}\left(\frac{d}{2\sqrt{2\sigma}}\right) \\
= \frac{1}{2} \text{erfc}\left(\frac{d}{2\sqrt{n_0}}\right) \\
= \frac{1}{2} \text{erfc}\left(\frac{\sqrt{E}}{\sqrt{2n_0}}\right)   
\]

(30.25)

\( \sigma^2 = n_0/2 \) for AWGN, and \( d = \sqrt{2E} \).

Table 30.1 summarises the performance of BPSK, orthogonal BFSK, and BASK signals in the presence of AWGN.

<table>
<thead>
<tr>
<th>Modulation</th>
<th>( d )</th>
<th>( P_e = \frac{1}{2} \text{erfc}\left(\frac{d}{2\sqrt{2\sigma}}\right) )</th>
<th>Relative (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPSK</td>
<td>( \sqrt{4E} )</td>
<td>( \frac{1}{2} \text{erfc}\left(\frac{\sqrt{E}}{n_0}\right) )</td>
<td>0</td>
</tr>
<tr>
<td>Orthogonal BFSK</td>
<td>( \sqrt{2E} )</td>
<td>( \frac{1}{2} \text{erfc}\left(\frac{\sqrt{E}}{2n_0}\right) )</td>
<td>-3</td>
</tr>
<tr>
<td>BASK</td>
<td>( \sqrt{E} )</td>
<td>( \frac{1}{2} \text{erfc}\left(\frac{\sqrt{E}}{4n_0}\right) )</td>
<td>-6</td>
</tr>
</tbody>
</table>

Table 30.1 Performance of various modulation systems.

Also, we can express the performance of these modulation systems in terms of the average signal energy \( \overline{E} \), where

\[
\overline{E} = E/2 
\]

(30.26)

for BASK signals,
Signal Space Analysis of BASK, BFSK, BPSK, and QAM on Mac

\[ \overline{E} = E \]  

(30.27)

for orthogonal BFSK signals, and

\[ \overline{E} = E \]  

(30.28)

for BPSK signals. This is summarised in Table 30.2.

<table>
<thead>
<tr>
<th>Modulation</th>
<th>( \overline{E} )</th>
<th>( P_e )</th>
<th>Relative (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPSK</td>
<td>( E )</td>
<td>( \frac{1}{2} \ erfc(\sqrt{\frac{E}{n_0}}) = \frac{1}{2} \ erfc(\sqrt{\frac{\overline{E}}{n_0}}) )</td>
<td>0</td>
</tr>
<tr>
<td>Orthogonal BFSK</td>
<td>( E )</td>
<td>( \frac{1}{2} \ erfc(\sqrt{\frac{E}{2n_0}}) = \frac{1}{2} \ erfc(\sqrt{\frac{\overline{E}}{2n_0}}) )</td>
<td>-3</td>
</tr>
<tr>
<td>BASK</td>
<td>( E/2 )</td>
<td>( \frac{1}{2} \ erfc(\sqrt{\frac{E}{4n_0}}) = \frac{1}{2} \ erfc(\sqrt{\frac{\overline{E}}{2n_0}}) )</td>
<td>-3</td>
</tr>
</tbody>
</table>

Table 30.2 Performance of various binary modulation systems in terms of the average signal energy.

Error Performance of \( M \)-ary QAM

Consider the \((M=16)\)-ary quadrature amplitude modulation (QAM) signal constellation diagram of Figure 30.4 as an example.

**Figure 30.4** 16-ary QAM signal constellation diagram.

The average signal energy of an \( M \)-ary QAM signal is

\[ \overline{E} = \frac{1}{M} \sum_{i=0}^{M-1} E_i \]  

(30.29)

where \( E_i \) is the energy of \( s_i \). For \( M = 16 \), we have

\[ \overline{E} = \frac{1}{16} \left\{ 4 \times \left( \frac{d_2}{4} + \frac{d_2}{4} \right) + 8 \times \left( \frac{9d_2^2}{4} + \frac{d_2^2}{4} \right) + 4 \times \left( \frac{9d_2^2}{4} + \frac{9d_2^2}{4} \right) \right\} \]

\( \{5,6,9,10\}\)  \( \{1,2,4,11,13,14,7,8\}\)  \( \{0,3,12,15\}\)
\( d = \frac{\sqrt{2E}}{5} \) 

(30.30)

and so

\[ d = \frac{\sqrt{2E}}{5} \]  

(30.31)

To find the probability of error, it is simpler to first find the probability \( P_c \) of correct detection. It is apparent from Figure 30.4 that we can group signal points into 3 classes and compute \( P_c \) based on the 3 decision regions as shown in Figure 30.5.

**Figure 30.5** Decision regions.

Let \( n_1 \) and \( n_2 \) be additive white Gaussian noise samples with two-sided power spectral density of \( n_0/2 \), zero mean and fixed variance \( \sigma^2 = n_0/2 \) along the \( \phi_1 \)-axis and \( \phi_2 \)-axis, respectively.

**Case 1 - four inner signal points \((s_5, 6, 9, 10)\):**

In the presence of AWGN, the probability of correct detection given that \( s_i, i = 5, 6, 9, 10 \), is transmitted is

\[
P(C/s_i) = P(-d/2 < n_1 < d/2) \times P(-d/2 < n_2 < d/2) \\
= P(-d/2 < n_1 < d/2) \times P(-d/2 < n_1 < d/2) \\
= p \times p
\]

(30.32)

where

\[
p = P(-d/2 < n_1 < d/2)
\]

\[
= \int_{-d/2}^{d/2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\phi_1^2}{2\sigma^2}} d\phi_1
\]

\[
= 2 \int_{0}^{d/2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\phi_1^2}{2\sigma^2}} d\phi_1
\]

(30.33)
Let \( y = \frac{\phi_1}{\sqrt{2\sigma}} \). Then \( y^2 = \frac{\phi_1^2}{2\sigma^2} \) and \( dy = \frac{d\phi_1}{\sqrt{2\sigma}} \). Substituting \( y \) and \( dy \) into equation (30.33), we get

\[
p = 2 \int_0^{2\sqrt{2\sigma}} \frac{1}{\sqrt{\pi}} e^{-y^2} dy
\]

\[
= \text{erf}(\frac{d}{2\sqrt{2}\sigma})
\]

\[
= \text{erf}(\frac{d}{2\sqrt{n_0}})
\]

(30.34)

where the error function is

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\alpha^2} d\alpha
\]

(30.35)

and \( \sigma^2 = n_0/2 \) for AWGN.

**Case 2 - four corner signal points \((s_0,3,12,15)\):**

In the presence of AWGN, the probability of correct detection given that \( s_i \), \( i = 0, 3, 12, 15 \), is transmitted is

\[
P(C/s_i) = P(-\infty < n_1 < d/2) \times P(-d/2 < n_2 < \infty)
\]

\[
= P(-d/2 < n_1 < \infty) \times P(-d/2 < n_2 < \infty)
\]

\[
= P(-d/2 < n_1 < \infty) \times P(-d/2 < n_1 < \infty)
\]

\[
= r \times r
\]

(30.36)

where

\[
r = P(-d/2 < n_1 < \infty)
\]

\[
= \int_{-d/2}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\phi_1^2}{2\sigma^2}} d\phi_1
\]
\[ \int_{-d/2}^{d/2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\phi_1^2}{2\sigma^2}} \, d\phi_1 + \int_{0}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\phi_1^2}{2\sigma^2}} \, d\phi_1 = 0.5p + 0.5 \]  

(30.37)

**Case 3 - eight edge signal points** (\(s_1, 2, 4, 7, 8, 11, 13, 14\)):

In the presence of AWGN, the probability of correct detection given that \(s_i\), \(i = 1, 2, 4, 7, 8, 11, 13, 14\), is transmitted is

\[
P(C/s_i) = P(-d/2 < n_1 < d/2) \times P(-d/2 < n_2 < \infty)
\]

\[
= P(-d/2 < n_1 < d/2) \times P(-d/2 < n_1 < \infty)
\]

\[
= p \times r
\]

(30.38)

Let \(P(s_i)\) be the probability of sending \(s_i\), for \(i = 0, 1,..., 15\). Thus, the total probability of correct detection is

\[
P_c = \sum_{i=0}^{M-1=15} P(s_i) \times P(C/s_i)
\]

(30.39)

With equally likely transmission of \(s_i\), we have

\[
P_c = 4 \left( \frac{1}{16} \, p \times p \right) + 4 \left( \frac{1}{16} \, r \times r \right) + 8 \left( \frac{1}{16} \, p \times r \right)
\]

\[
= \frac{1}{16} \left[ 4 \, p^2 + 4 \left( \frac{p}{2} + \frac{1}{2} \right)^2 + 8p \left( \frac{p}{2} + \frac{1}{2} \right) \right]
\]

\[
= \frac{(3p + 1)^2}{16}
\]

(30.40)

The probability of error is

\[
P_e = 1 - P_c
\]

\[
= 1 - \frac{(3p + 1)^2}{16} = \frac{4^2 - (3p + 1)^2}{16}
\]

\[
= \frac{3(1-p)(5+3p)}{16}
\]

30.11
For normal operation, $p$ approaches 1 and $P_e << 1$, and then

$$P_e = \frac{3(1-p)8}{16} = \frac{3}{2} (1 - p) \quad (30.41)$$

Substituting $p = \text{erf} \left( \frac{d}{2\sqrt{n_0}} \right)$ and $d = \sqrt{\frac{25}{E}}$ into equation (30.41), we have

$$P_e \approx \frac{3}{2} \left[ 1 - \text{erf} \left( \frac{\sqrt{E}}{10n_0} \right) \right] = \frac{3}{2} \text{erfc} \left( \frac{\sqrt{E}}{10n_0} \right) \quad (30.42)$$

**Example 30.1**

Consider the 4-ary QAM signal constellation diagram as shown in Figure 30.6.

**Figure 30.6** 4-ary QAM signal constellation diagram.

The average signal energy is

$$\bar{E} = \frac{1}{4} \left[ 4 \times \left( \frac{d^2}{4} + \frac{d^2}{4} \right) \right] = \frac{d^2}{2}$$

and

$$d = \sqrt{\frac{25}{E}}.$$  

From our previous analysis of 16-ary QAM signals, the probability of correct detection in 4-ary QAM signals given that $s_i$, $i = 0, 1, 2, 3$, is transmitted is

$$P(C/s_i) = P(-\infty < n_1 < d/2) \times P(-d/2 < n_2 < \infty) \times P(-d/2 < n_1 < \infty) \times P(-d/2 < n_1 < \infty) = r \times r$$

where
\[ r = 0.5p + 0.5 \]

\[ p = \text{erf}\left( \frac{d}{2\sqrt{2}\sigma} \right) = \text{erf}\left( \frac{d}{2\sqrt{n_0}} \right) \]

and \( \sigma^2 = n_0/2 \) for AWGN.

The total probability of correct detection is

\[ P_c = \sum_{i=0}^{M-1=3} P(s_i) P(C/s_i) \text{ a priori probability} \]

With equally likely transmission of \( s_i \), we have

\[ P_c = 4 \left( \frac{1}{4} r \times r \right) = \left( \frac{p}{2} + \frac{1}{2} \right)^2 = \frac{(1+p)^2}{4} \]

The probability of error is

\[ P_e = (1 - P_c) = 1 - \frac{(1+p)^2}{4} = \frac{(1-p)(3+p)}{4} \]

For normal operation, \( p \) approaches 1 and \( P_e \ll 1 \), and then

\[ P_e \approx \frac{4(1-p)}{4} = (1-p) \] (30.43)

Substituting \( p = \text{erf}\left( \frac{d}{2\sqrt{n_0}} \right) \) and \( d = \sqrt{2E} \) into equation (30.43), we have

\[ P_e \approx 1 - \text{erf}\left( \sqrt{\frac{E}{2n_0}} \right) = \text{erfc}\left( \sqrt{\frac{E}{2n_0}} \right). \]
References


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**Figure 30.1** BASK signal constellation diagram and the conditional probability density functions associated with the signals.

**Figure 30.2** BPSK signal constellation diagram and the conditional probability density functions associated with the signals.

**Figure 30.3** Orthogonal BFSK signal constellation diagram and the conditional probability density functions associated with the signals.
Figure 30.4 16-ary QAM signal constellation diagram.

Figure 30.5 Decision regions.
Figure 30.6 4-ary QAM signal constellation diagram.